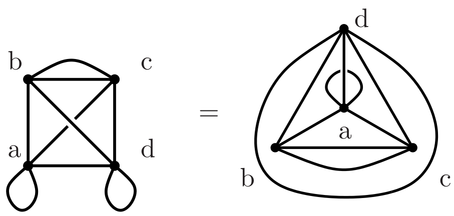
Graphs

Definition. A *graph* G is a finite set of *vertices* $V(G)$ and a finite set $E(G)$ of unordered pairs (x, y) of vertices $x, y \in V(G)$ called *edges.*

A graph may have *loops* (x, x) and *multiple edges* when a pair (x, y) appears in $E(G)$ several times. Pictorially we represent the vertices by points and edges by lines connecting the corresponding points. Topologically a graph is a 1-dimensional *cell complex* with $V(G)$ as the set of 0-cells and $E(G)$ as the set of 1-cells. Here are two pictures representing the same graph.



$V(G) = \{a, b, c, d\}$

$E(G) = \{(a, a), (a, b), (a, c), (a, d), (b, c), (b, c), (b, d), (c, d), (d, d)\}$

$\chi_\Delta(q) = q(q-1)(q-2) = q^3 - 3q^2 + 2q$

Chromatic polynomial $\chi_G(q)$.

A *coloring* of G with q colors is a map $\varkappa : V(G) \to \{1, \ldots, q\}$. A coloring $\varkappa$ is *proper* if for any edge e: $\varkappa(v_1) \neq \varkappa(v_2)$, where v_1 and v_2 are the endpoints of e.

Definition 1. $\chi_G(q)$:= # *of proper colorings of G in q colors.*

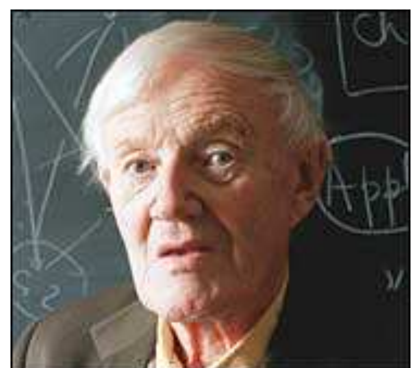
Properties (Definition 2).

$\chi_G = \chi_{G-e} - \chi_{G/e}$;

$\chi_{G_1 \sqcup G_2} = \chi_{G_1} \cdot \chi_{G_2}$, for a disjoint union $G_1 \sqcup G_2$;

$\chi_\bullet = q$.

Stanley's theorem. *For a graph G with n vertices,* $(-1)^n \chi_G(-1) =$ # *of acyclic orientations of G.*



Tutte polynomial $T_G(x, y)$.

Definition 1.

$T_G = T_{G-e} + T_{G/e}$	*if e is neither a bridge nor a loop ;*
$T_G = xT_{G/e}$	*if e is a bridge ;*
$T_G = yT_{G-e}$	*if e is a loop ;*
$T_{G_1 \sqcup G_2} = T_{G_1 \cdot G_2} = T_{G_1} \cdot T_{G_2}$	*for a disjoint union* $G_1 \sqcup G_2$ *and a one-point join* $G_1 \cdot G_2$;

$T_\bullet = 1$.

Properties.

$T_G(1, 1)$	is the number of spanning trees of G ;
$T_G(2, 1)$	is the number of spanning forests of G ;
$T_G(1, 2)$	is the number of spanning connected subgraphs of G ;
$T_G(2, 2) = 2^{\lvert E(G) \rvert}$	is the number of spanning subgraphs of G ;
$\chi_G(q) = q^{k(G)}(-1)^{r(G)} T_G(1-q, 0)$;	

$\chi_\Delta(-1) = -6$

Definition 2.

Let
- F be a graph;
- $v(F)$ be the number of its vertices;
- $e(F)$ be the number of its edges;
- $k(F)$ be the number of connected components of F;
- $r(F) := v(F) - k(F)$ be the *rank* of F;
- $n(F) := e(F) - r(F)$ be the *nullity* of F;

$= x(x+1)+y$

$q\,T_{\Delta}(1-q,0) = q(1-q)(2-q) = q(q-1)(q-2) = \chi_{\Delta}(q)$

$$T_G(x,y) \ := \sum_{F\subseteq E(G)} (x-1)^{r(G)-r(F)}(y-1)^{n(F)}$$

Dichromatic polynomial $Z_G(q,v)$ (Definition 3).

Let $Col(G)$ denote the set of colorings of G with q colors.

$$Z_G(q,v) \ := \sum_{\varkappa\in Col(G)} (1+v)^{\#\text{ edges colored not properly by }\varkappa}$$

Properties .

$Z_G = Z_{G-e} + vZ_{G/e}$;

$Z_{G_1\sqcup G_2} = Z_{G_1}\cdot Z_{G_2}$, for a disjoint union $G_1\sqcup G_2$;

$Z_{\bullet} = q$;

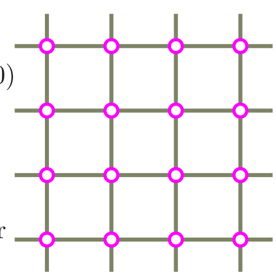
$Z_G(q,v) = \sum_{F\subseteq E(G)} q^{k(F)}v^{e(F)}$;

$\chi_G(q) = Z_G(q,-1)$;

$Z_G(q,v) = q^{k(G)}v^{r(G)}T_G(1+qv^{-1},1+v)$;

$T_G(x,y) = (x-1)^{-k(G)}(y-1)^{-v(G)}Z_G((x-1)(y-1),y-1)$.

Potts model in statistical mechanics (Definition 4).



Potts model (C.Domb 1952); $q=2$ the Ising model (W.Lenz, 1920)

Let G be a graph.

Particles are located at vertices of G. Each particle has a *spin*, which takes q different values . A *state*, $\sigma\in\mathcal{S}$, is an assignment of spins to all vertices of G. Neighboring particles interact with each other only is their spins are the same.
The energy of the interaction along an edge e is $-J_e$ (*coupling constant*). The model is called *ferromagnetic* if $J_e>0$ and *antiferromagnetic* if $J_e<0$.

Energy of a state σ (*Hamiltonian*),

$$H(\sigma) = -\sum_{(a,b)=e\in E(G)} J_e\ \delta(\sigma(a),\sigma(b)).$$

Boltzmann weight of σ:

$$e^{-\beta H(\sigma)} = \prod_{(a,b)=e\in E(G)} e^{J_e\beta\delta(\sigma(a),\sigma(b))} = \prod_{(a,b)=e\in E(G)}\Bigg(1+(e^{J_e\beta}-1)\delta(\sigma(a),\sigma(b))\Bigg),$$

where the *inverse temperature* $\beta=\frac{1}{\kappa\ T}$, T is the temperature, $\kappa = 1.38\times 10^{-23}$ joules/Kelvin is the *Boltzmann constant*.

The Potts partition function (for $x_e := e^{J_e\beta}-1$)

$$Z_G(q,x_e) \ := \sum_{\sigma\in\mathcal{S}} e^{-\beta H(\sigma)} = \sum_{\sigma\in\mathcal{S}}\prod_{e\in E(G)}(1+x_e\delta(\sigma(a),\sigma(b)))$$

Properties of the Potts model Probability of a state σ: $P(\sigma) := e^{-\beta H(\sigma)}/Z_G$.

Expected value of a function $f(\sigma)$:

$$\langle f\rangle := \sum_\sigma f(\sigma)P(\sigma) = \sum_\sigma f(\sigma)e^{-\beta H(\sigma)}/Z_G\ .$$

Expected energy: $\langle H\rangle = \sum_\sigma H(\sigma)e^{-\beta H(\sigma)}/Z_G = -\frac{d}{d\beta}\ln Z_G\ .$

Fortuin—Kasteleyn'1972: $Z_G(q,x_e) = \sum_{F\subseteq E(G)} q^{k(F)}\prod_{e\in F} x_e\ ,$

where $k(F)$ is the number of connected components of the spanning subgraph F.
$Z_G = Z_{G\setminus e} + x_e Z_{G/e}\ .$

Stanley's chromatic symmetric function. [St1]

$$\boxed{X_G(x_1,x_2,\dots) := \sum_{\substack{\varkappa:V(G)\to\mathbb{N}\\ \text{proper}}}\ \prod_{v\in V(G)} x_{\varkappa(v)}}$$

Example.

$$\begin{aligned} X_{\bullet\!\!-\!\!\bullet} &= \widehat{x_1x_1} + x_1x_2 + x_1x_3 + \dots\\ &\quad x_2x_1 + \widehat{x_2x_2} + x_2x_3 + \dots\\ &\quad x_3x_1 + x_3x_2 + \widehat{x_3x_3} + \dots\\ &\quad\ \ \vdots \qquad\ \vdots \qquad\qquad \ddots\\ &= p_1^2 - p_2,\ \text{where}\end{aligned}$$

$p_m := \sum_{i=1}^{\infty} x_i^m$ is the *power function basis* for the space of symmetric functions.

Symmetric Stanley's acyclicity theorem deals with the expression of X_G in terms of another basis, the basis of *elementary symmetric functions.*

$$\begin{aligned} e_0 &:= 1;\\ e_1 &:= x_1 + x_2 + \dots + x_n + \dots = p_1;\\ e_2 &:= (x_1x_2 + x_1x_3 + \dots + x_1x_n + \dots) + (x_2x_3 + \dots + x_2x_n + \dots) + \dots + (x_{n-1}x_n + \dots) + \dots;\\ e_3 &:= x_1x_2x_3 + \dots\dots + x_{n-2}x_{n-1}x_n + \dots;\\ &\ \ \vdots \end{aligned}$$

In general, $\boxed{e_k := \sum_{1\le j_1<j_2<\dots<j_k} x_{j_1}\dots x_{j_k}}$.

In particular, $p_2 = e_1^2 - 2e_2$, $X_{\bullet\!\!-\!\!\bullet} = e_1^2 - (e_1^2 - 2e_2) = 2e_2$, and $X_{K_k} = k!e_k$.

Theorem. [St1, Theorem 3.3] *Let $X_G = \sum c_{\lambda_1,\lambda_2,\dots,\lambda_s} e_{\lambda_1}e_{\lambda_2}\dots e_{\lambda_s}$ be the expression of X_G in terms of elementary symmetric functions. (Note that $\lambda_1+\lambda_2+\dots+\lambda_s = \#$ of vertices of G.) Then for every s, $\sum c_{\lambda_1,\lambda_2,\dots,\lambda_s} = \#$ of acyclic orientations of G with exactly s sinks.*

For example, the graph $G = \bullet\!\!-\!\!\bullet$ has 2 acyclic orientations with exactly one sink $\bullet\!\!\to\!\!\bullet$ and $\bullet\!\!\leftarrow\!\!\bullet$, because $X_{\bullet\!\!-\!\!\bullet} = 2e_2$.

Chromatic polynomial of signed graphs.

There are two chromatic polynomials of signed graphs [Za].

A *q-coloring* of a signed G is a map $\varkappa : V(G) \to \{-q,-q+1,\dots,-1,0,1,\dots,q-1,q\}$. A q-coloring $\varkappa$ is *proper* if for any edge e with the sign ε_e: $\varkappa(v_1) \neq \varepsilon\varkappa(v_2)$, where v_1 and v_2 are the endpoints of e.

The signed chromatic polynomial $\chi_G^{\neq 0}(2q)$ is a specialization of Y_G obtained by substitution $x_i = 1$ for $|i| \leq q$ and $x_i = 0$ for $|i| \geq q$. This is the same substitution as $p_{a,b} = \lambda = 2q$.

References

[Ch] S. Chmutov, *B-symmetric chromatic function of signed graphs*, https://people.math.osu.edu/chmutov.1/talks/2020/slides-Moscow.pdf Video is available at https://www.youtube.com/watch?v=khA7rP84sYY

[Ra] R. Raghavan, *A Symmetric Chromatic Function for Signed Graphs*, https://people.math.osu.edu/chmutov.1/wor-gr-su19/Rushil-Raghavan-MIGHTY_10192019.pdf

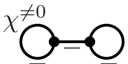
[St1] R. Stanley, *A symmetric function generalization of the chromatic polynomial of a graph*, Advances in Math. **111**(1) (1995) 166–194.

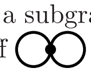
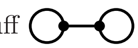
[Za] T. Zaslavsky, *Signed graph coloring*, Discrete Mathematics **39**(2) (1982) 215–228.

Definition.
$\chi_G(2q+1) :=$ *# of proper q-colorings of G.*
$\chi^{\neq 0}_G(2q) :=$ *# of proper q-colorings of G which take nonzero values.*

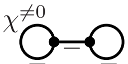
Properties.

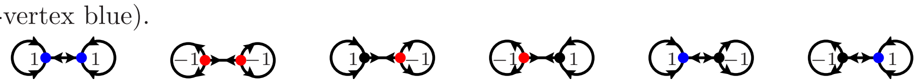
- $\chi_G(\lambda)$ is a polynomial function of $\lambda = 2q+1 > 0$;
- $\chi^{\neq 0}_G(\lambda)$ is a polynomial function of $\lambda = 2q > 0$;
- $\chi_G(\lambda) = \chi_{G-e}(\lambda) - \chi_{G/e}(\lambda)$;
- $\chi^{\neq 0}_G(\lambda) = \chi^{\neq 0}_{G-e}(\lambda) - \chi^{\neq 0}_{G/e}(\lambda)$;
- $\chi_{G_1 \sqcup G_2} = \chi_{G_1} \cdot \chi_{G_2}$ and $\chi^{\neq 0}_{G_1 \sqcup G_2} = \chi^{\neq 0}_{G_1} \cdot \chi^{\neq 0}_{G_2}$ for a disjoint union $G_1 \sqcup G_2$;
- $\chi_\emptyset = 1$.

Example. $\chi^{\neq 0}$  $(2q) = 2q(2q-1)$.

There are two tricky issues in Zaslavky's acyclicity theorem. The first one is the notion of a cycle. A subset S of edges of a sign graphs is called *balanced* if for every circuit in S the product of the signs of edges of the circuit is equal to 1. A *cycle* of a signed graph G is a subgraph of one the following 3 types: 1) a balanced circuit, 2) a subdivision of a tight handcuff  with both circuits to be unbalanced, and 3) a subdivision of a loose handcuff  with both circuits to be unbalanced. The second issue is a notion of orientation. An *orientation* of an edge is a pair of arrows on its half-edges which are coherent for positive edges and not coherent for negative edges. An orientation of a sign graph is *acyclic* if every cycle contains either a source or a sink. An orientation of a sign graph is *compatible* with a coloring c if for every positive edge the color of it arrow-head is greater or equal to the color of it arrow-tail and for every negative edge the sum of colors of its ends is not negative (resp. not positive) for the arrows pointed towards the ends (resp. away from the end).

Theorem. [Za, Theorem 3.5] *Let $q \in \mathbb{N}$ and G be a signed graph with n vertices. The the number of compatible pairs of acyclic orientations of G and colorings $V(G) \to \{-q, -q+1, \ldots, -1, 1, \ldots, q-1, q\}$ is equal to $(-1)^n \chi^{\neq 0}_G(-2q)$.*

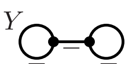
Example. For $q = 1$ and the graph above we have $\chi^{\neq 0}$  $(-2) = 6$. Here are 6 compatible pairs of acyclic orientations and colorings $V(G) \to \{-1, 1\}$ (we mark the source-vertex red and the sink-vertex blue).



Note that the first two colorings are proper and the last four are improper.

<u>***B*-symmetric chromatic function of signed graphs**</u>. [Ra, Ch]

$$\boxed{Y_G(\ldots, x_{-2}, x_{-1}, x_1, x_2, \ldots\ldots) := \sum_{\substack{\varkappa : V(G) \to \mathbb{Z}\setminus\{0\} \\ \text{proper}}} \prod_{v \in V(G)} x_{\varkappa(v)}}$$

Example. Y 

$$\begin{aligned} &= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\quad x_{-2}(\ldots x_{-2} + x_{-1} + x_1 + \widehat{x_2} + \ldots) \\ &\quad x_{-1}(\ldots x_{-2} + x_{-1} + \widehat{x_1} + x_2 + \ldots) \\ &\quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &= p_{1,0}^2 - p_{1,1}, \text{ where} \end{aligned}$$

$p_{a,b} := \sum_{i \in \mathbb{Z}\setminus\{0\}} x_i^a x_{-i}^b$ are the *signed power functions*.